

The Barber Paradox: on its Paradoxicality and its Relationship to Russell's Paradox

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Abstract

The Barber paradox is often introduced as a popular version of Russell's paradox, though some experts have denied their similarity, even calling the Barber paradox a pseudoparadox. In the first part of the paper, I demonstrate mainly that in the standard (Quinean) definition of a paradox the Barber paradox is a clear-cut example of a non-paradox. Despite some outward similarities, it differs radically from Russell's paradox. I also expose many other differences. In the second part of the paper, I examine a probable source of the paradoxicality of the Barber paradox, which is found in a certain ambivalence in terms of meaning. The two different readings of the crucial phrase yield distinct existential assumptions which produce the paradoxical conclusion.

One of the paradoxical aspects of the Barber paradox (*BP*) is that it is not a paradox, though many people still think it is (including some philosophers that address paradoxes, e.g., Nicholas Rescher, [14], pp. 143–147). It is also paradoxical that its authorship is often attributed to Bertrand Russell (e.g. [8]), even though he did not invent it,¹ and even warned against it as a false analogy to Russell's paradox (*RP*).² Finally, it is also somewhat paradoxical that despite what I have previously written, I will suggest a way to construe the BP as a real paradox.

This paper begins with an exposition of both RP and the BP, mainly showing the ways in which they are not similar. We will see that, according to the standard Quinean definition of a paradox, the BP is simply not a paradox at all. Then, I will attempt to reveal a possible source of the paradoxicality of the BP, which is based on a confusion of meaning. The degree of paradoxicality will be studied for a modified BP.

¹The probable author of the BP is the Austrian philosopher Ernst Mally. This guess originates from Alonzo Church ([3], p. 106), who mentioned it in his short review of a paper by Johann Mokre. Mokre attributed the Postman paradox ("Postabholer", "Postzusteller", [11], p. 82) to Mally: Consider a villager who volunteers to bring the mail to all those who do not go to get their own mail – does he bring the mail to himself? ([11], p. 89).

²Russell mentioned the BP explicitly as a false analogy to RP in his large study *The Philosophy of Logical Atomism* published in *Monist* 1918–19 ([15], p. 101).

1 (Dis)similarities between Russell's paradox and the Barber paradox

I will operate from the standard *definition of a paradox* adopted from Quine's seminal article *The Ways of Paradox*, [13]:³ a paradox is an *argument* whose conclusion contradicts a widely shared opinion or, as I will call it, a *naïve theory* which forms a usually hidden premise of the argument. To illustrate, the naïve theory of truth is, in the form of a certain general claim, a hidden premise of the Liar paradox. A paradox as an argument is usually reconstructed from its formulation, which is often rather incomplete, leaving the formation of the paradox to the imagination of a reader. The solution of a paradox consists in a (justified) refutation of the problematic premise (or naïve theory) in question or in a (justified) refutation of some inferential step applied within the argument. All these claims could be made more accurate, but let's accept them as they are.⁴

RP, discovered by Russell at the very beginning of the 20th century, is the crucial modern paradox (for a recent reflection of RP cf., e.g., [9]). Since a detailed description of RP does not lie within the scope of this paper, I am only going to state a few of its features which are needed for its comparison with the BP.

The hidden premise of RP is *naïve set theory*. More precisely, it is an opinion incorporated within it, viz. the claim that a set can be defined by any condition (predicate).⁵ This is usually articulated as the (naïve, unrestricted) *Axiom of Comprehension* (or Abstraction)

$$\forall F \exists s \forall x ((x \in s) \leftrightarrow F(x)).$$

In other words, for any condition F there is a set s of individuals satisfying the condition F .

As is well known, Russell challenged this opinion with an attempt to define set R , Russell's set, with the help of the expression:

the set of all and only those sets which are not members of themselves

which can be formalized as $\{s \mid s \notin s\}$. Its core is the condition $s \notin s$, which is quite legitimate in naïve set theory. However, Russell showed that

³The Quinean definition has been popularized by Mark Sainsbury in his well-known introductory book *Paradoxes*, in which he defines a paradox as: "an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises" ([16], p. 1).

⁴Some doubts about the Quinean definition have been recently expressed, e.g., by William G. Lycan [10], yet they will not be considered here.

⁵Though there is a difference between predicates and conditions, we can abstract from it in this text.

there are complications with this. If we assume that R contains itself, we are then forced to conclude that it does not contain itself because R contains only those sets which are not members of themselves; on the other hand, if we assume that R does not contain itself, we must conclude that it does contain itself; in both cases there is a contradiction. To avoid the paradox, approaches such as Russell's or Church's type theory and Zermelo-Fraenkel's set theory, which is one of the critical theories concerning sets, do not allow an unrestricted version of the principle according to which any condition (predicate) defines a set.

A formulation similar to the following one is used to induce the BP:⁶ In a certain village, there is a barber who shaves all and only those villagers who do not shave themselves – does the barber shave himself? If we assume that the barber does not shave himself, we conclude that he must also shave himself; if we assume that the barber does shave himself, we conclude that he cannot; in both cases, a contradiction is engendered.

Many sum up the case of the barber by simply stating that such a barber, *just like* set R , cannot exist. (We will focus on the existential issues in the next section.) Another similarity between the BP and RP is based on the structural similarity between the key expressions from the formulation of the paradox, e.g. the description

the individual who shaves all and only those individuals who do not shave themselves.

However, there are also some important *dissimilarities*. Firstly, there is a huge difference between the sets which should be determined by the two crucial expressions. In the case of the BP an *empty set* is specified – no individual is its member. In other words, the formula

$$\forall y(\text{Sh}(x, y) \leftrightarrow \sim \text{Sh}(y, y))$$

is false because the condition in question requires somebody who shaves himself precisely if he does not shave himself. However, no possible common value of variables x and y can have both R and non- R to itself. This led Joseph Fraser Thomson ([17], p. 104) to the right conclusion that

⁶One of the BP's popular variants is the *Catalogue paradox* formulated by the French mathematician Ferdinand Gonseth ([4], p. 254): In a certain library, there is a catalogue which lists all (and only those) catalogues which do not list themselves – does this catalogue list itself? Gonseth (p. 255) described this paradox as having both set and Barber variants. A very similar version is the *bibliography of all bibliographies* (cf., e.g., [14], p. 144, where nearly a dozen such phrases are offered). Another popular version is the unnamed paradox concerning *secretaries* which was published by L.S. Johnston ([7], p. 474) as a deliberately popular variant of the RP: There is a club C having a regulation according to which only a secretary of a club to which she is not eligible to join can become a member of this club C – is the secretary of C eligible to join C ? In German-speaking regions (e.g., [11], p. 82, [1], p. 8) one can also often meet the *Self-murderer's paradox* ("Selbstmörder").

$$\sim \exists x \forall y (\text{Sh}(x, y) \leftrightarrow \sim \text{Sh}(y, y))$$

is a theorem of classical logic.⁷

In the case of RP, on the other hand, the empty set is not specified. As documented in the above considerations: *no set* is specified at all.

This important ‘empty set/no set’ difference has a connection with the following dissimilarity. If one ascertains that no such village barber can possibly exist, one gives up the unsupported empirical assumption that there is such a barber. The conclusion that there really is no such barber is highly acceptable. On the other hand, the non-existence of Russell’s set R leads to the questioning of the reasonable mathematical assumption that such a set as set R should exist. The conclusion that there really is no such set R is thus difficult to accept. This explanation was already offered by Quine ([13], p. 12) and we will partly return to such considerations in the next section.

Employing the standard definition of a paradox, the *crucial dissimilarity* between the two paradoxes consists in the fact that the premise of RP as an argument is naïve set theory, while the BP as an argument does not contain the set theory as its premise. Moreover, the BP contains *no naïve theory*, thus there is nothing which could be contradicted by the conclusion (remember “para-doxa”). Hence, the BP is *not a paradox* at all. This difference between the two arguments can be easily recognized in the fact that RP has the effect of refuting of naïve set theory, while the BP does not lead us to the refutation of any naïve theory.

Vigorous refutations of the BP as an analogy of RP can be found at various places in the specialized literature, though the explanation just given is not, as far as I know, presented there. One of the first dismissals, important from the viewpoint of its future acceptance by other writers, is by Kurt Grelling ([5], p. 481), who wrote in negative response to Chaïm Perelman ([12], p. 205), who considered the BP to be a proper logical paradox. Grelling even wrote that the solution to the paradox (“joke”) can perhaps be a useful exercise for young students of logic. It is then not surprising that the BP, and its version by Johnston ([7]), was already labelled a *pseudoparadox* by Church ([2], p. 157).

⁷Note that $\sim \exists s \forall s' ((s \subseteq s') \leftrightarrow \sim (s' \subseteq s'))$ is a theorem of second-order logic. Some who claim that RP and the BP are analogous (cf. [6]) can be suspected of the confusion of \in with \subseteq – only \in gives rise to a paradox.

2 The source of the paradoxicality of the Barber paradox studied with reference to its modifications

When Quine ([13], p. 12) emphatically wrote “we had never positively believed in such a barber,” he missed, I think, something important about the problem. It is also a bit inconvenient to just throw the BP away onto the pseudoparadox dump. Since the BP intuitively seems to be a paradox, any serious approach to paradoxes should investigate this fact. Such an attempt is made in this section, in which we encounter the phenomenon of the degree of paradoxicality.

Firstly, I am going to show how to build a proper paradox from the usual formulation of the BP. One has to explicitly add a certain problematic premise, i.e. that every property, including

being an x such that for every y , x shaves y if and only if y does not shave y ,

has at least one bearer – that an individual exists that instantiates the property. This is contradicted by the conclusion that there can be no such individual. Of course, the theory of the instantiation of properties is hardly generally held. Nevertheless, this has no effect on what I intend to show, namely that the BP would be a proper paradox with such an additional premise.⁸

An explanation as to why some think the BP is a “simpler version” of RP might be that they insert the additional premise into their reconstruction of the BP. The theory that every property has an instance is correlative with the view that every condition (predicate) defines a set, thus the internal similarity of RP and the BP, which we noted as lacking in a preceding section, would be preserved. Since the theory of the instantiation of properties is unlikely to have adherents, this explanation of the BP’s paradoxicality can be sidelined.

The supplemented BP is apparently akin to the Horned Man paradox. The Horned Man is not a paradox at all, unless one holds the rather uncommon premise that one really has everything that one has not lost. Thus, whether something is considered a paradox depends partly on the degree of recognition of the problematic naïve theory. The paradoxicality of the Horned Man paradox is very low, while the paradoxicality of the recent Hypergame paradox ([20]) is rather high. The paradoxicality of RP is smaller than that of the Hypergame paradox, since it has nearly become a logician’s

⁸Note that one formulation of a paradox may give rise to more than one paradox as an argument, or a paradox and a non-paradox.

platitude, to use Quine’s own wording,⁹ that some conditions do not define a set.

Now let us focus on the source of the paradoxicality of the BP as it is normally conceived. For my findings it is convenient to recall that during ordinary human communication speakers assume that we are speaking about existing things. To repeat Strawson’s well-known example, when somebody says that his children went to sleep, we usually presuppose that he has some children. The formulation of the BP even begins with the assumption that such a village barber exists, or can exist. After examining the paradox, however, we concluded that there could be no such barber. We are thus putting our finger on the most probable reason for the BP’s paradoxicality, which consists in a hidden *ambiguity of meaning*. The two meanings of the crucial expression imply distinct existential assumptions, and this is the root of the contradiction.

Due to *irreflexive reading* of the BP’s crucial expression – “shaves all and only those who do not shave themselves” –, the barber shaves *all others*, everybody except himself.¹⁰ This is perhaps a natural reading because an artisan or specialist is often understood to be somebody who does something for others. However, the question “Does he shave himself?” led hearers to speculate about the other meaning of the expression: its *reflexive reading*, due to which the artisan can also serve himself likewise.

I have said that the paradoxicality of the BP originates in the fact that the following sentences are contradictory:

(P) *There exists a barber that shaves all and only those who do not shave themselves,*

(C) *There does not exist a barber that shaves all and only those who do not shave themselves.*

However, premise P and conclusion C are only contradictory provided they are both read in the reflexive sense. With the reflexive reading, sentence C is *logically* (i.e. *necessarily*) *true*. If P is really contradictory to C, P is *logically false*. It thus implies a specific existential assumption as regards the barber, namely that *no such barber can possibly exist*.

Nevertheless, it is rather unnatural to understand P just in the reflexive sense. Especially if one assumes, as the reader of the paradox does, that such a barber can exist. P is thus understood rather as P’:

(P’) *There exists a barber that shaves all and only those others who do not shave themselves.*

⁹“One man’s antinomy can be another man’s veridical paradox, and one man’s veridical paradox can be another man’s platitude” ([13], p. 12).

¹⁰The suggestion to replace the quantifier “all” with the quantifier “all others”, using thus an addition of $(x \neq y)$ in the antecedent of the biconditional which was formally stated above, was already made by Johan F. A. K. van Benthem ([18], p. 52).

With this irreflexive reading, P – i.e. P' – is *contingently true*. Its relevant specific existential assumption thus differs from that of the reflexive reading: *such a barber can exist by a contingent chance*.

To sum up, the paradoxicality of the BP consists in the fact that the premise P , which is assumed to be possibly true (being rather P'), turned out to be necessarily false, because the meaning of the crucial expression has switched.

One might perhaps utilize this observation concerning P and P' to say that the BP is analogous to RP after all. In this case, the following argument would be advanced as the explicit form of the BP:

(P'') *Possibly, there is a barber who shaves all and only those who do not shave themselves,*

(C'') *Necessarily, there is no barber who shaves all and only those who do not shave themselves.*

The explicit form of RP would be quite analogous to this, but mentioning set membership instead of a barber and shaving.

Despite this one similarity between the modified BP and RP, a principal difference still remains between the two paradoxes because one must consider the relevant conditions of being true. The possible existence of such a barber should occur within the empirical world of material beings and the internal consistency or inconsistency involved in the description of the barber is a plain matter. However, Russell's set R should exist within the realm of immaterial mathematical entities, in which a consistency question is usually a *condicio sine qua non*. The naïve theory of sets also postulated a realm of sets in which a set containing (*inter alia*) R would be an *inconsistent multiplicity* (to use Cantor's own phrase, cf. [19], p. 114), just as the set M of entirely all the sets there are – including the subsets of M 's powerset, which is impossible with Cantor's theorem. Note that nothing remotely similar can be said about the Barber's *pseudoparadox*.

3 Conclusions

In the first main section of this paper, I have argued that, according to the standard definition of a paradox, the BP is not a paradox at all. In its usual form, the BP harbours no naïve theory (in the form of a problematic premise) which could be contradicted by its conclusion. The BP thus has only an incidental resemblance to RP. Various similarities and dissimilarities between both paradoxes have been discussed throughout the whole paper.

In the second main section, I have focused on the problem of the degree of paradoxicality. A probable source of the admitted paradoxicality of the BP consists in the ambiguity in the meaning of the verb “shave”, and thus the

crucial phrase of the BP, which results in different conclusions concerning the possible existence of a certain barber. We have seen then that the nature of these existential assumptions of the BP differs significantly from that of the assumptions involved in the background of RP.

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